

# CALOGERO-MOSER PAIRS AND THE AIRY AND BESSEL BISPECTRAL INVOLUTIONS

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**ABSTRACT.** Explicit formulae are given for the Airy and Bessel bispectral involutions, in terms of Calogero-Moser pairs. Hamiltonian structure of the motion of the poles of the operators is discussed.

## 1. INTRODUCTION

This paper follows upon the study of the Airy bispectral involution made in [KR]. There we gave an analogue, for arbitrary rank, of the rank-one bispectral involution developed by Wilson [W1]. Recently, [W2], Wilson has established a relationship between the rank-one bispectral involution and the complex analogue of the Calogero-Moser phase space. This relationship leads to explicit formulae for the Baker function and the corresponding involution, which make many important features manifest. As shown below, similar results hold for bispectral algebras obtained from generalized Airy and Bessel operators.

Given a positive integer  $n$ , define  $C_n$  to be the quotient, under conjugation by  $GL(n, \mathbb{C})$ , of the space of pairs (*Calogero-Moser pairs*) of  $n \times n$  complex matrices,  $(P, Q)$ , such that

$$(1.1) \quad \text{rank}([P, Q] - I) = 1 .$$

This is the complex analogue of the definition in [KKS], in which  $P$  and  $Q$  are taken to be hermitian, and  $I$  is replaced by  $iI$ . Define  $Gr^{ad}$  (see [W1, W2]) to be the subspace of the Sato grassmannian ([DJKM, Sa, SW]) corresponding to Krichever data whose spectral curve is rational and unicursal (no nodes). The Baker function for such data is always of the form  $e^{xz}p(x, z)$ , where  $p(x, z)$  is rational and separable. By separable we mean its denominator is a product  $q(z)\tau(x)$ . Wilson has proved

**Theorem 1.1** ([W2], **Thm. 5.1**). *There is a one-to-one correspondence*

$$(1.2) \quad \bigcup_{n=0}^{\infty} C_n \leftrightarrow Gr^{ad} ,$$

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such that a point  $W \in Gr^{ad}$  corresponds to a point  $(P, Q)$  if and only if its Baker function,  $\psi_W$  is given by

$$(1.3) \quad \psi_W(x, z) = e^{xz} \text{Det}(I - (zI - Q)^{-1}(xI - P)^{-1}) .$$

Remark: The “generic” rational unicursal curve has simple cusps. These correspond to  $Q$  with distinct eigenvalues. In the hermitian case this is automatic. An important aspect of Wilson’s theorem is the observation that in the complex case, nonsemisimple  $Q$  correspond to nonsimple cusps, or in physical terms, collision of the Calogero-Moser particles, which are now moving in  $\mathbb{C}$  rather than  $\mathbb{R}$ .

The spectral algebra  $\mathcal{R}_W$  is the algebra of differential operators  $L(x, \partial)$  such that  $L\psi_W = f(z)\psi_W$  for some function  $f(z)$ . Wilson’s result proves that the spectral algebra of any point  $W \in Gr^{ad}$  is bispectral in the sense of [DG]. That is, the spectral algebra of  $\psi_W(z, x)$  is also nontrivial.

It proves much more, for it says that  $\psi_W(z, x)$  is also a Baker function, namely the one corresponding to  $(Q^T, P^T)$ . The involution  $(P, Q) \mapsto (Q^T, P^T)$  is clearly antisymplectic with respect to the symplectic form

$$(1.4) \quad \omega = \text{tr}(dP \wedge dQ) .$$

This symplectic structure is an important example of “unreduction” [KKS]. Namely, the Calogero-Moser hierarchy [AMM] is a completely integrable hamiltonian system defined on the quotient space  $C_n$ . The hamiltonians are rather complicated in the reduced variables, but on the level of matrices  $(P, Q)$  they are given simply by the hamiltonians are  $h_n = \text{tr}(P^n)$ . Moreover, the involution  $((P, Q)) \longrightarrow ((Q^T, P^T))$  is the linearizing map for the Calogero-Moser particle system [AMM, KKS]. Thus, Wilson’s result gives the best proof that this linearizing map and the bispectral involution are one and the same (cf. [Ka]). As we shall see, the corresponding involutions, in terms of an auxilliary monic polynomial  $\rho(t)$ , are given in the Airy and Bessel cases respectively by

$$(1.5) \quad (P, Q) \mapsto (P^T, \rho(P^T) - Q^T) ,$$

$$(1.6) \quad (P, Q) \mapsto ((QP\rho(QP)^{-1}Q)^T, (Q^{-1}\rho(QP))^T) .$$

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## 2. THE AIRY CASE

The rank of a commutative algebra  $\mathcal{R}$  of ordinary differential operators is defined to be the greatest common divisor of the orders of its elements. The true rank of  $\mathcal{R}$  is

the rank of its centralizer [LP, PW]. For instance, fix a monic  $r$ th order polynomial  $\rho(t)$ , and define

$$(2.1) \quad L_{\text{Ai}} = \rho(\partial) - x \quad (\text{Airy})$$

Then  $\mathbb{C}[L_{\text{Ai}}]$  has true rank  $r$  [KR]. Moreover, this algebra is bispectral. Indeed, for any  $f \in \ker(L_{\text{Ai}})$ , define

$$(2.2) \quad \hat{f}_{\text{Ai}}(x, z) = f(x + z) .$$

Then

$$(2.3) \quad L_{\text{Ai}}(\hat{f}_{\text{Ai}}) = z\hat{f}_{\text{Ai}} , \quad L_{\text{Ai}}(z, \partial_z)(\hat{f}_{\text{Ai}}) = x\hat{f}_{\text{Ai}} .$$

To generalize Wilson's formula to this case, introduce a *Baker functional*,  $\Psi_{\text{Ai}, W}$ , defined on  $\ker L_{\text{Ai}}$ , of the form

$$(2.4) \quad \Psi_{\text{Ai}}(f) = \sum_{i=0}^{r-1} k_i(x, z) \partial^i(\hat{f}_{\text{Ai}}) .$$

To specify its properties, it is useful to introduce the dual description of  $Gr^{ad}$  (cf. [W1, KR]). As Wilson shows, every point of  $W \in Gr^{ad}$  arises from a homogeneous, finite dimensional space of finitely supported distributions in the complex plane. That is, there should be complex numbers  $\lambda_1, \dots, \lambda_n$  and polynomials over  $\mathbb{C}$ ,  $\ell_1, \dots, \ell_n$ , such that

$$(2.5) \quad W = \frac{1}{q(z)} \{ p(z) \in \mathbb{C}[z] \mid c_i(p) = 0 \} ,$$

where

$$(2.6) \quad q(z) = \prod_i (z - \lambda_i)$$

and

$$(2.7) \quad c_i = \delta_{\lambda_i} \circ \ell(\partial_z) .$$

Now define  $\Psi_{\text{Ai}, W}$  by the following properties. Let  $C$  denote the span of  $c_1, \dots, c_n$ .

Property 1a: The functions  $q(z)k_i(x, z)$  are polynomial in  $z$ .

Property 2a: For all  $f \in \ker(L)$ ,  $q_C(z)\Psi_{\text{Ai}, W}(f)$  is annihilated by all  $c \in C$ .

Property 3a:  $\lim_{z \rightarrow \infty} (k_0, \dots, k_{r-1}) = (1, 0, \dots, 0)$ .

Fixing the operator  $L_{\text{Ai}}$ , set

$$(2.8) \quad \vec{k}_W = \begin{pmatrix} k_0 \\ \vdots \\ k_{r-1} \end{pmatrix}.$$

It is important to remark that the map  $C \mapsto W$  is not one-to-one. The equivalence relation on  $C$  induced by this map is the one generated by

$$(2.9) \quad C + \mathbb{C}\delta_\lambda \sim C \circ (z - \lambda) .$$

In particular, the properties of  $\Psi_{\text{Ai},W}$  depend only on  $W$  and not on the representative  $C$ .

Choose a space of conditions  $C$  defining  $W$ , and set  $k_i(x, z) = \frac{1}{q_C(z)} \sum_j k_{i,j}(x) z^j$ . Consider the differential operator

$$(2.10) \quad K_{\text{Ai},C} = \sum_{i,j} k_{i,j}(x) \partial^i L_{\text{Ai}}^j .$$

Defining

$$(2.11) \quad \vec{f}_{\text{Ai}}(x, z) = (\hat{f}_{\text{Ai}}, \partial(\hat{f}_{\text{Ai}}), \dots, \partial^{r-1}(\hat{f}_{\text{Ai}})) ,$$

one has

$$(2.12) \quad K_{\text{Ai},C}(\hat{f}_{\text{Ai}}) = q_C(z) \vec{f}_{\text{Ai}} \cdot \vec{k}_W .$$

The asymptotics of  $\vec{k}_W$  imply that  $K_{\text{Ai},C}$  is a monic operator of order  $rn$ . Property 2a implies that  $K_{\text{Ai},C}$  annihilates  $c(\hat{f}_{\text{Ai}})$  for all the distributions  $c \in C$  and all  $f \in \text{Ker}(L_{\text{Ai}})$ . Thus  $K_{\text{Ai},C}$  is unique operator with the two properties just stated. It then follows (cf. [KR]) that for any polynomial  $p \in R_W$ , the pseudodifferential operator  $M_p = K_{\text{Ai},C} p(L_{\text{Ai}}) K_{\text{Ai},C}^{-1}$  is a differential operator, and

$$(2.13) \quad M_p(K_{\text{Ai},C}(\hat{f}_{\text{Ai}})) = p(z) K_{\text{Ai},C}(\hat{f}_{\text{Ai}})$$

for all  $f \in \text{Ker}(L_{\text{Ai}})$ . Note that  $M_p$  could also have been obtained by conjugating  $p(L_{\text{Ai}})$  by the monic 0th order pseudodifferential operator  $K_{\text{Ai},C} q(L_{\text{Ai}})^{-1}$ . The latter operator is independent of the space  $C$  representing  $W$ , and is the analogue of the Sato operator in this theory.

Thus one has a rank  $r$  commutative algebra of differential operators

$$(2.14) \quad \mathcal{R}_{\text{Ai},W} = \{ M_p \mid p \in R_W \} ,$$

with an  $r$ -dimensional space of eigenfunctions  $\Psi_{\text{Ai},W}(f)$ ,  $f \in \text{Ker}(L)$ . To accomodate the usual normalization of spectral algebras, namely that the subprincipal symbol should vanish, take  $\rho$  of the form  $\rho(t) = t^r + O(t^{r-2})$ .

The task is to obtain a formula for  $\vec{k}_W$  in terms of the matrices  $P$  and  $Q$  corresponding to the point  $W$ . The following lemma expresses property 2a in terms of covariant differentiation of  $\vec{k}_W$ . Set

$$(2.15) \quad B_{Ai}(x, z) = \begin{bmatrix} 0 & \dots & 0 & x+z \\ 1 & \ddots & & a_1 \\ 0 & \ddots & \vdots & a_2 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{bmatrix},$$

where the  $a_i$ 's are the coefficients of  $\rho$ , and set

$$(2.16) \quad \nabla_{Ai} = \frac{\partial}{\partial z} + B_{Ai}(x, z).$$

**Lemma 2.1.** *Let  $c$  be a distribution of the form  $c = \delta_\lambda \circ p(\partial_z)$ . Let  $\vec{g} = \begin{bmatrix} g_0(x, z) \\ \vdots \\ g_{r-1}(x, z) \end{bmatrix}$  be a vector of polynomials in  $z$  with coefficients in  $\mathbb{C}(x)$ . Then  $c(\vec{f}_{Ai} \cdot \vec{g}) = 0$  for all  $f \in \text{Ker}(L)$  if and only if  $\vec{g}$  is annihilated by  $\delta_\lambda \circ p(\nabla_{Ai})$ .*

*Proof.* Given  $f \in \text{Ker}(L)$ ,

$$(2.17) \quad \partial_z(\vec{f}_{Ai}) = \vec{f}_{Ai} \cdot B_{Ai}(x, z).$$

Thus, for all  $j$ ,

$$(2.18) \quad \delta_\lambda \circ \partial_z^j(\vec{f}_{Ai} \cdot \vec{g}) = \vec{f}_{Ai}(x, \lambda) \cdot (\partial_z + B_{Ai}(x, \lambda))^j(\vec{g}).$$

This proves the lemma, since there is no differential equation of order less than  $r$  satisfied by  $f(x + \lambda)$  for all  $f \in \text{Ker}(L)$ .  $\square$

Now consider the involutions on each  $C_n$  defined by

$$(2.19) \quad \begin{aligned} C_n &\xrightarrow{\beta_{Ai}} C_n \\ ((P, Q)) &\mapsto ((\hat{P}, \hat{Q})), \quad \text{where} \\ \hat{P} &= P^T; \quad \hat{Q} = \rho(P^T) - Q^T. \end{aligned}$$

**Theorem 2.2.** *Let  $W \in Gr^{ad}$  correspond to a point  $((P, Q)) \in C_n$ . Let*

$$(2.20) \quad [P, Q] = I - w_1 w_2^T$$

*where  $w_1$  and  $w_2$  are column vectors. For  $j = 0, \dots, r-1$ , let*

$$(2.21) \quad \rho_j(t) = t^{r-1-j} - \sum_{i=j+1}^{r-2} a_i t^{i-1-j}.$$

*Then the components of  $\vec{k}_W$  are*

$$(2.22) \quad k_j(x, z) = \delta_{0,j} - w_2^T (zI - Q)^{-1} \rho_j(P) (x - \hat{Q}^T)^{-1} w_1.$$

*Proof.* It suffices to consider the generic case, in which  $W$  is defined by conditions  $c_i = \delta_{\lambda_i} \circ (\partial - \alpha_i)$ ,  $i = 1, \dots, n$ , for distinct  $\lambda$ 's. Set

$$(2.23) \quad \gamma_i = \alpha_i - \sum_{j \neq i} \frac{1}{\lambda_i - \lambda_j}.$$

Then  $W$  corresponds under Wilson's theorem to the Calogero-Moser pair

$$(2.24) \quad Q = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$$(2.25) \quad P = \begin{bmatrix} \frac{\gamma_1}{\lambda_2 - \lambda_1} & \frac{1}{\lambda_1 - \lambda_2} & \cdots & \frac{1}{\lambda_1 - \lambda_n} \\ \vdots & & \ddots & \vdots \\ \frac{1}{\lambda_n - \lambda_1} & \cdots & & \gamma_n \end{bmatrix}.$$

Let  $(e_j)_{j=1, \dots, r}$  be the standard basis for  $\mathbb{C}^r$ . Then

$$(2.26) \quad \vec{k}_W = e_1 + \sum_{i=1}^n \frac{\vec{v}_i(x)}{(z - \lambda_i)},$$

for some vectors  $\vec{v}_i(x) \in \mathbb{C}^r$ . Applying  $\delta_{\lambda_i} \circ (\partial_z + B_{Ai}(x, \lambda_i) - \alpha_i)$  to  $q(z)\vec{k}_W$ , one obtains the set of equations

$$(2.27) \quad \begin{aligned} 0 = & B_{Ai}(x, \lambda_i) \vec{v}_i(x) \prod_{\ell \neq i} (\lambda_i - \lambda_\ell) + \\ & \prod_{\ell \neq i} (\lambda_i - \lambda_\ell) e_1 + \sum_{m=1}^n \vec{v}_m(x) \sum_{\ell \neq m} \prod_{s \neq \ell, m} (\lambda_i - \lambda_s) \\ & - \alpha_i \vec{v}_i(x) \prod_{\ell \neq i} (\lambda_i - \lambda_\ell), \end{aligned}$$

$i = 1, \dots, n$ . Dividing by  $\prod_{\ell \neq i} (\lambda_i - \lambda_\ell)$  and using (2.23),

$$(2.28) \quad e_1 = -B_{\text{Ai}}(x, \lambda_i) \vec{v}_i(x) + \gamma_i \vec{v}_i(x) - \sum_{\ell \neq i} \frac{\vec{v}_\ell(x)}{\lambda_i - \lambda_\ell}.$$

Let  $(u_i)_{i=1, \dots, n}$  be the standard basis for  $\mathbb{C}^n$ , and let

$$(2.29) \quad v(x) = \sum u_i \otimes \vec{v}_i(x) \in \mathbb{C}^n \otimes \mathbb{C}^r.$$

Let  $w = \sum u_i$ . Write  $B_{\text{Ai}}(x, \lambda_i) = B_{\text{Ai}}(x, 0) + \lambda_i \Delta$ , where

$$(2.30) \quad \Delta = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Then (2.28) is encoded as a single equation

$$(2.31) \quad w \otimes e_1 = (-I \otimes B(x) + P \otimes I - Q \otimes \Delta) v(x).$$

Altogether, (2.26) becomes

$$(2.32) \quad \vec{k}_W = e_1 - (w^T \otimes I) \circ ((zI - Q)^{-1} \otimes I) \circ A^{-1}(w \otimes e_1),$$

where

$$(2.33) \quad A = I \otimes B_{\text{Ai}}(x, 0) - P \otimes I + Q \otimes \Delta \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^r).$$

Thinking of  $\mathbb{C}^n \otimes \mathbb{C}^r$  as  $rn$ -tuples in blocks of length  $n$ ,

$$(2.34) \quad w \otimes e_1 = \left[ \begin{array}{c} 1 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{array} \right] \Bigg\}^n$$

and

$$(2.35) \quad A = \begin{bmatrix} -P & 0 & \cdots & 0 & xI + Q \\ I & -P & & & a_1 I \\ 0 & I & \ddots & & \vdots \\ \vdots & 0 & & & a_{r-1} I \\ 0 & 0 & 0 & I & -P \end{bmatrix}.$$

One checks quite easily that

$$(2.36) \quad A \begin{bmatrix} \rho_0(P) \\ \vdots \\ \vdots \\ \rho_{r-1}(P) \end{bmatrix} = \begin{bmatrix} xI + Q - \rho(P) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Then

$$(2.37) \quad \begin{aligned} & (w^T \otimes I) \circ ((zI - Q)^{-1} \otimes I) \circ A^{-1}(w \otimes e_1) \\ &= \begin{bmatrix} w^T & & 0 \\ & \ddots & \\ 0 & & w^T \end{bmatrix} \begin{bmatrix} (zI - Q)^{-1} & & 0 \\ & \ddots & \\ 0 & & (zI - Q)^{-1} \end{bmatrix} \begin{bmatrix} \rho_0(P) \\ \vdots \\ \vdots \\ \rho_{r-1}(P) \end{bmatrix} (xI + Q - \rho(P))^{-1} w. \\ &= \begin{bmatrix} w^T(zI - Q)^{-1}\rho_0(P)(xI - \hat{Q}^T)^{-1}w \\ \vdots \\ w^T(zI - Q)^{-1}\rho_{r-1}(P)(xI - \hat{Q}^T)^{-1}w \end{bmatrix}. \end{aligned}$$

This proves the theorem, since

$$(2.38) \quad [P, Q] = I - ww^T.$$

□

**Corollary 2.3.** *For any generalized Airy operator  $L_{Ai}$ , and for all  $W \in Gr^{ad}$ ,*

$$(2.39) \quad \vec{k}_W(z, x) = \vec{k}_{\beta_{Ai}(W)}(x, z) .$$

*In particular, the algebra  $\mathcal{R}_{Ai, W}$ , (2.14), is bispectral, with an  $r$ -dimensional space of joint eigenfunctions  $\Psi_{Ai, W}(f)$ ,  $f \in Ker(L_{Ai})$ .*

*Proof.* Formula (2.39) follows immediately from (2.22), for if  $[P, Q] = I - w_1 w_2^T$ , then

$$(2.40) \quad [\hat{P}, \hat{Q}] = [P^T, \rho(P^T) - Q^T] = [P, Q]^T = I - w_2 w_1^T .$$

The rest of the corollary is immediate. □



### 3. THE BESSEL CASE

The Bessel case works in much the same way. Consider again a polynomial  $\rho(t)$ , now normalized so that  $a_{r-1} = \binom{r}{2}$ . Set

$$(3.1) \quad L_{\text{Be}} = x^{-r} \rho(D) \quad (\text{Bessel}) ,$$

where

$$(3.2) \quad \partial = \frac{d}{dx} , \quad D = x\partial .$$

Consider  $\text{Ker}(L_{\text{Be}} - 1)$ , which should now be thought of as a sheaf rather than a space. For  $f \in \text{Ker}(L_{\text{Be}} - 1)$ , define

$$(3.3) \quad \hat{f}_{\text{Be}}(x, z) = f(xz) .$$

Then

$$(3.4) \quad L_{\text{Be}}(\hat{f}_{\text{Be}}) = z^r \hat{f}_{\text{Be}} , \quad L_{\text{Be}}(z, \partial_z)(\hat{f}_{\text{Be}}) = x^r \hat{f}_{\text{Be}} .$$

Assume now that the matrix  $Q$  is invertible. Define, as the analogue of (2.11),

$$(3.5) \quad \vec{f}_{\text{Be}}(x, z) = (\hat{f}_{\text{Be}}, D(\hat{f}_{\text{Be}}), \dots, D^{r-1}(\hat{f}_{\text{Be}})) .$$

Then

$$(3.6) \quad D_z(\vec{f}_{\text{Be}}) = \vec{f}_{\text{Be}} B_{\text{Be}}(x^r, z^r) ,$$

where

$$(3.7) \quad B_{\text{Be}}(x, u) = \begin{bmatrix} 0 & \dots & 0 & a_0 + xu \\ 1 & \ddots & & a_1 \\ 0 & \ddots & \vdots & a_2 \\ \vdots & \ddots & & \vdots \\ & & 0 & a_{r-2} \\ 0 & \dots & 0 & 1 & a_{r-1} \end{bmatrix} .$$

Accordingly, one expects a Baker functional of the form

$$(3.8) \quad \Psi_{\text{Be}, W}(f) = \vec{f}_{\text{Be}} \cdot \vec{k}_W(x^r, z^r) , \quad f \in \text{Ker}(L_{\text{Be}} - 1) .$$

To state the properties of  $\Psi_{\text{Be}, W}$ , introduce the functions

$$(3.9) \quad \mu(x, z) = (x^r, z^r) ; \quad \nu(z) = z^r .$$

Denote by  $\nu^*$  the action of  $\nu$  on the space of finitely supported distributions in  $\mathbb{C}^*$ . Let

$$(3.10) \quad \nabla_{\text{Be}} = D_z + \frac{1}{r} B_{\text{Be}}(x, z) .$$

Given a distribution  $c = \delta_\lambda \circ p(D)$ , define

$$(3.11) \quad c_{\nabla_{\text{Be}}} = \delta_\lambda \circ p(\nabla_{\text{Be}}) ,$$

acting on vector valued functions of  $z$ .

**Lemma 3.1.** *Let  $c$  be a distribution of the form  $c = \delta_\lambda \circ p(D)$ . Let  $\vec{g} = \begin{bmatrix} g_0(x, z) \\ \vdots \\ g_{r-1}(x, z) \end{bmatrix}$  be a vector of polynomials in  $z$  with coefficients in  $\mathbb{C}(x)$ . Then  $c(\vec{f}_{\text{Be}} \cdot \mu^*(\vec{g})) = 0$  for all  $f \in \text{Ker}(L_{\text{Be}} - 1)$  if and only if  $\nu^*(c)_{\nabla_{\text{Be}}}(\vec{g}) = 0$ .*

*Proof.* By virtue of the identity

$$(3.12) \quad D \circ \nu^* = \nu^* \circ rD ,$$

one has

$$(3.13) \quad \nu^*(c) = \delta_{\lambda^r} \circ p(rD) .$$

By (3.6),

$$\begin{aligned} c \circ \vec{f}_{\text{Be}} \cdot \mu^*(\vec{g}) &= \delta_\lambda \circ \vec{f}_{\text{Be}} \cdot p(D + B_{\text{Be}}(x^r, z^r)) \circ \mu^*(\vec{g}) \\ &= \vec{f}_{\text{Be}}(x, \lambda) \cdot \delta_\lambda \circ \mu^* \circ p(rD + B_{\text{Be}}(x, z))(\vec{g}) \\ &= \vec{f}_{\text{Be}}(x, \lambda) \cdot \nu_x^* \circ \delta_{\lambda^r} \circ p(r\nabla_{\text{Be}})(\vec{g}) \\ (3.14) \quad &= \vec{f}_{\text{Be}}(x, \lambda) \cdot \nu_x^* \circ \nu^*(c)_{\nabla_{\text{Be}}}(\vec{g}) , \end{aligned}$$

where  $\nu_x$  is  $\nu$  acting in the  $x$ -variable. The lemma now follows as in lemma 2.1.  $\square$

In light of the preceding lemma, it makes sense to impose the following properties on  $\Psi_{\text{Be}, W}$ .

Property 1b: The functions  $q_C(z)k_i(x, z)$  are polynomial in  $z$ .

Property 2b: Let  $C'$  be any space of distributions such that  $\nu^*(C') = C$ . Then for all  $f \in \text{Ker}(L_{\text{Be}} - 1)$ ,  $q_C(z)\Psi_{\text{Be}, W}(f)$  is annihilated by all  $c \in C'$ .

Property 3b:  $\lim_{z \rightarrow \infty} \vec{k}_W = e_1$ .

As in the Airy case, one reconstructs a differential operator  $K_{\text{Be}, C}$ , but now

$$(3.15) \quad K_{\text{Be}, C}(\hat{f}_{\text{Be}}) = q_C(z^r)\vec{f}_{\text{Be}} \cdot \vec{k}_W(x^r, z^r) .$$

Then for any polynomial  $p \in R_W$ , the pseudodifferential operator  $M_p = K_{\text{Be},C} p(L_{\text{Be}}) K_{\text{Be},C}^{-1}$  is a differential operator, and

$$(3.16) \quad M_p(K_{\text{Be},C}(\hat{f}_{\text{Be}})) = p(z^r) K_{\text{Be},C}(\hat{f}_{\text{Be}})$$

for all  $f \in \text{Ker}(L_{\text{Be}} - 1)$ . Define  $\mathcal{R}_{\text{Be},W}$  to be the algebra of the all the  $M_p$ 's.

Everything now proceeds as before. Assume that the matrix  $Q$  is invertible. We have  $n$  distributions  $c_i = \delta_{\lambda_i} \circ (\partial_z - \alpha_i)$ . Note that  $\delta_{\lambda_i} \circ (\partial_z - \alpha_i) = \frac{1}{\lambda_i} \delta_{\lambda_i} \circ (D_z - \lambda_i \alpha_i)$ . Thus, according to lemma 3.1, property 2b imposes the  $n$  conditions

$$(3.17) \quad 0 = \delta_{\lambda_i} \circ (\partial_z + \frac{1}{r\lambda_i} B_{\text{Be}}(x, \lambda_i) - \alpha_i)(q_C(z) \vec{k}_W(x, z)) .$$

Setting

$$(3.18) \quad \vec{k}_W = e_1 + \sum_{i=1}^n \frac{\vec{v}_i(x)}{(z - \lambda_i)} ,$$

one now finds

$$(3.19) \quad e_1 = -\frac{1}{r\lambda_i} B_{\text{Be}}(x, \lambda_i) \vec{v}_i(x) + \gamma_i \vec{v}_i(x) - \sum_{\ell \neq i} \frac{\vec{v}_\ell(x)}{\lambda_i - \lambda_\ell} .$$

This time,

$$(3.20) \quad \frac{1}{ru} B_{\text{Be}}(x, u) = \frac{1}{ru} \Delta_1 + \frac{x}{r} \Delta_2 ,$$

where

$$(3.21) \quad \Delta_1 = \begin{bmatrix} 0 & \cdots & 0 & a_0 \\ 1 & \ddots & \vdots & \vdots \\ 0 & \ddots & 0 & \vdots \\ & \ddots & & \\ 0 & \cdots & 0 & 1 & a_{r-1} \end{bmatrix} ,$$

$$(3.22) \quad \Delta_2 = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & \vdots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{bmatrix} .$$

Thus

$$(3.23) \quad w \otimes e_1 = -(\frac{x}{r} I \otimes \Delta_2 - P \otimes I + \frac{1}{r} Q^{-1} \otimes \Delta_1) v(x) .$$

Then

$$(3.24) \quad \vec{k}_W = e_1 - (w^T \otimes I) \circ ((zI - Q)^{-1} \otimes I) \circ A^{-1}(w \otimes e_1),$$

where  $A$  is now given in block matrix form by

$$(3.25) \quad A = \begin{bmatrix} -P & 0 & \cdots & 0 & \frac{x}{r}I + \frac{a_0}{r}Q^{-1} \\ \frac{1}{r}Q^{-1} & \ddots & & \vdots & \frac{a_1}{r}Q^{-1} \\ 0 & \ddots & & 0 & \vdots \\ \vdots & \ddots & & -P & \frac{a_{r-2}}{r}Q^{-1} \\ 0 & \cdots & 0 & \frac{1}{r}Q^{-1} & -P + \frac{a_{r-1}}{r}Q^{-1} \end{bmatrix}.$$

One obtains the following result.

**Theorem 3.2.** *Let  $W \in Gr^{ad}$  correspond to a point  $((P, Q)) \in C_n$ . Let*

$$(3.26) \quad [P, Q] = I - w_1 w_2^T$$

where  $w_1$  and  $w_2$  are column vectors. Writing the  $r$ th order Bessel operator  $L_{Be}$  in the form  $L_{Be} = x^{-r}\rho(D)$ , let

$$(3.27) \quad \rho_j(t) = t^{r-1-j} - \sum_{i=j+1}^{r-1} a_i t^{i-1-j}$$

for  $j = 0, \dots, r-1$ . Then the components of  $\vec{k}_W$  are

$$(3.28) \quad k_j(x, z) = \delta_{0,j} - r w_2^T (zI - Q)^{-1} \rho_j(rQP)(x - \hat{Q}^T)^{-1} w_1,$$

where

$$(3.29) \quad \hat{Q} = (Q^{-1} \rho(rQP))^T.$$

*Proof.* One checks now that with  $A$  given by (3.25),

$$(3.30) \quad A \begin{bmatrix} r\rho_0(rQP) \\ \vdots \\ r\rho_{r-1}(rQP) \end{bmatrix} = \begin{bmatrix} xI - Q^{-1}\rho(rQP) \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

The result then follows as in theorem 2.2.  $\square$

Theorem 3.2 suggests the definition

$$(3.31) \quad \begin{aligned} \hat{P} &= \hat{Q}^{-1} P^T Q^T \\ &= (QP \rho(QP)^{-1} Q)^T. \end{aligned}$$

Then

$$\begin{aligned}
 \hat{\hat{Q}} &= (\hat{Q}^{-1} \rho(r \hat{Q} \hat{P}))^T \\
 &= (\hat{Q}^{-1} \rho(r P^T Q^T))^T \\
 (3.32) \quad &= \rho(r Q P) \rho(r Q P)^{-1} Q = Q ,
 \end{aligned}$$

and

$$\begin{aligned}
 \hat{\hat{P}} &= \hat{\hat{Q}}^{-1} \hat{\hat{P}}^T \hat{\hat{Q}}^T \\
 (3.33) \quad &= Q^{-1} Q P (\hat{Q}^{-1})^T \hat{Q}^T = P .
 \end{aligned}$$

Moreover,

$$\begin{aligned}
 [\hat{P}, \hat{Q}] &= [\hat{Q}^{-1} P^T Q^T, \hat{Q}] \\
 &= \hat{Q}^{-1} P^T Q^T \hat{Q} - P^T Q^T \\
 &= Q^T (\rho(r Q P)^{-1})^T (Q P)^T \rho(r Q P)^T (Q^{-1})^T - P^T Q^T \\
 (3.34) \quad &= Q^T P^T - P^T Q^T = [P, Q]^T .
 \end{aligned}$$

So again one has an involution (densely defined) on  $C_n$ ,

$$\begin{aligned}
 C_n &\xrightarrow{\beta_{Be}} C_n \\
 (P, Q) &\mapsto (\hat{P}, \hat{Q}) , \quad \text{where} \\
 (3.35) \quad \hat{P} &= \hat{Q}^{-1} P^T Q^T ; \quad \hat{Q} = (Q^{-1} \rho(r Q P))^T .
 \end{aligned}$$

**Corollary 3.3.** *For any generalized Bessel operator  $L_{Be}$ ,*

$$(3.36) \quad \vec{k}_W(z, x) = \vec{k}_{\beta_{Be}(W)}(x, z) .$$

*In particular, the algebra  $\mathcal{R}_{Be, W}$  is bispectral, with a rank  $r$  joint eigensheaf  $\Psi_{Be, W}(f)$ ,  $f \in \text{Ker}(L_{Be} - 1)$ .*

#### 4. DYNAMICS

It is well-known that the Calogero-Moser particle system is a completely integrable hamiltonian system on the symplectic manifold  $C_n$ . The symplectic form is

$$(4.1) \quad \omega = \text{tr}(dP dQ) ,$$

and the hamiltonians are  $h_n = \text{tr}(P^n)$  (cf. [KKS]).

It is pleasing then that the Airy and Bessel involutions are antisymplectic on each  $C_n$ . In the Airy case,

$$(4.2) \quad \hat{P} = P^T ; \quad \hat{Q} = \rho(P^T) - Q^T ,$$

this follows from the fact that

$$(4.3) \quad \text{tr}(dP dP^n) = \sum_{i+j=n-1} \text{tr}(dP P^i dP P^j) = 0 .$$

(The basic trace identity for form-valued matrices is

$$(4.4) \quad \text{tr}(XY) = (-1)^{\deg(X)\deg(Y)} \text{tr}(YX) .)$$

The Bessel case is slightly more involved.

**Proposition 4.1.** *Let  $\sigma$  be a polynomial over  $\mathbb{C}$ . Let*

$$(4.5) \quad \hat{Q} = (Q^{-1}\sigma(QP))^T ; \quad \hat{P} = \hat{Q}^{-1}P^T Q^T .$$

*Then*

$$(4.6) \quad \text{tr}(d\hat{P}d\hat{Q}) = -\text{tr}(dP dQ) .$$

*Proof.* Let  $\tilde{Q} = Q^{-1}\sigma(QP)$ ,  $\tilde{P} = QP\sigma(QP)^{-1}Q = \hat{P}^T$ . Then

$$(4.7) \quad \text{tr}(d\hat{P}d\hat{Q}) = \text{tr}(d\tilde{P}d\tilde{Q}) .$$

Set  $R = QP$ . Then

$$(4.8) \quad \text{tr}(dP dQ) = d \text{tr}(Q^{-1}R dQ) ,$$

while

$$(4.9) \quad \begin{aligned} \text{tr}(d\tilde{P}d\tilde{Q}) &= d \text{tr}(R\sigma^{-1}Q d(Q^{-1}\sigma)) \\ &= d \text{tr}(R\sigma^{-1}d\sigma - R\sigma^{-1}dQ Q^{-1}\sigma) \\ &= d \text{tr}(R\sigma^{-1}d\sigma - Q^{-1}R dQ) . \end{aligned}$$

So it must be proved that

$$(4.10) \quad d \text{tr}(R\sigma^{-1}d\sigma) = 0 .$$

If  $\sigma = \sigma_1\sigma_2$  and  $\sigma_2$  commutes with  $R$ , then

$$(4.11) \quad \begin{aligned} d \text{tr}(R\sigma^{-1}d\sigma) &= d \text{tr}(R\sigma_2^{-1}\sigma_1^{-1}(d\sigma_1\sigma_2 + \sigma_1 d\sigma_2)) \\ &= d \text{tr}(R\sigma_1^{-1}d\sigma_1) + d \text{tr}(R\sigma_2^{-1}d\sigma_2) . \end{aligned}$$

Also,

$$(4.12) \quad d \text{tr}(\sigma^{-1}d\sigma) = -\text{tr}(\sigma^{-1}d\sigma\sigma^{-1}d\sigma) = 0 .$$

This last identity implies that one can replace  $R$  by  $R - (\text{const})I$  in (4.10), and reduce to the case that  $\sigma = R\sigma_1(R)$ ,  $\sigma_1$  polynomial.

Then by (4.11)

$$(4.13) \quad \begin{aligned} d \operatorname{tr}(R\sigma^{-1}d\sigma) &= d \operatorname{tr}(dR + R\sigma_1^{-1}d\sigma_1) \\ &= d \operatorname{tr}(R\sigma_1^{-1}d\sigma_1) . \end{aligned}$$

Since the result is obvious when  $\sigma$  is a constant, the proposition follows by induction on the degree of  $\sigma$ .  $\square$

Now introduce time dependence into the Baker functionals of the previous sections in a manner generalizing the standard procedure in the rank-one case [SW]. Fix a positive integer  $m$ . The time-dependent Baker function in the rank one case is the function

$$(4.14) \quad e^{xz+tz^m} p(x, z, t)$$

satisfying properties 1,2 and 3 of section 2 with a fixed space of conditions  $C$ . On the other hand, one may introduce time-dependence into the conditions by defining  $C_t = C \circ e^{tz^m}$ . Then the function  $e^{xz} p(x, z, t)$  satisfies properties 1,2 and 3 for the variable conditions  $C_t$ .

If  $c = \delta_\lambda \circ (\partial - \alpha)$ , then  $c \circ e^{tz^m} = \delta_\lambda \circ (\partial + tm\lambda^{m-1} - \alpha)$ . In other words, the flow  $C_t$  is seen on the level Calogero-Moser pairs as

$$(4.15) \quad Q_t = Q_0 ; \quad P_t = P_0 - tmQ^{m-1} .$$

This is the flow of the completely integrable hamiltonion

$$(4.16) \quad h_m = \operatorname{tr}(Q^m) .$$

This hamiltonian is the Calogero-Moser hamiltonian with the roles of  $P$  and  $Q$  reversed. Finally, the Baker function gives rise in a standard way to a solution of the KP-hierarchy, with poles in  $x$  the same as those of the Baker function. Thus, Wilson's formula makes it immediately clear that the poles in  $x$  of such a KP-solution move as a Calogero-Moser particle system (cf. [Kr, Sh, Ka]).

To carry this over to the Airy and Bessel cases, define  $\vec{k}_{W,t}$  to be the vector (2.8), for the variable space of conditions  $C_t$ . We are led to solutions of a subhierarchy of the KP-hierarchy, in the following way. Let  $K_{0,Ai}$  be the monic  $0^{th}$ -order pseudodifferential operator such that

$$(4.17) \quad K_{0,Ai} \partial^r K_{0,Ai}^{-1} = L_{Ai} .$$

Let  $\tilde{K}_t$  be the monic  $0^{th}$ -order pseudodifferential operator such that

$$(4.18) \quad \vec{f}_{Ai} \cdot \vec{k}_{W,t} = \tilde{K}_t(\hat{f}_{Ai}) .$$

Then the argument in [SW] shows that

$$(4.19) \quad \partial_t(\tilde{K})\tilde{K}^{-1} + (\tilde{K}L_{Ai}^m\tilde{K}^{-1})_- = 0 .$$

Now let

$$(4.20) \quad K_t = \tilde{K}_t K_{0, \text{Ai}} .$$

Then

$$(4.21) \quad \partial_t(K)K^{-1} + (K\partial^{rm}K^{-1})_- = 0 .$$

It now follows as in [SW] that the operator

$$(4.22) \quad M_t = K_t \partial K_t^{-1}$$

satisfies the  $rm^{th}$  term of the KP-hierarchy,

$$(4.23) \quad \partial_t(M) = [M_+^{rm}, M] .$$

From formula (2.22),  $M$  has coefficients in  $\mathbb{C}(t)[x, \frac{1}{\tau_t(x)}]$ , where

$$(4.24) \quad \tau_t(x) = \text{Det}(x - \hat{Q}_t) ,$$

$$(4.25) \quad \begin{aligned} \hat{Q}_t &= (\rho(P_t) - Q_t)^T \\ &= (\rho(P_0 - tmQ_0^{m-1}) - Q_0)^T . \end{aligned}$$

Thus we have constructed a solution  $M_t$ , of the  $rm^{th}$  term of the KP-hierarchy, whose poles in  $x$  move according to the completely integrable hamiltonian  $\text{tr}(\hat{Q}^m)$  on the Calogero-Moser phase space.

In the Bessel case, exactly the same analysis holds, except that  $M$  has coefficients in  $\mathbb{C}(t)[x, \frac{1}{x}, \frac{1}{\tau_t(x)}]$ . Thus  $M_t$  has a fixed pole at  $x = 0$ , with the motion of the remaining poles being governed by the hamiltonian  $\text{tr}(\hat{Q}^m)$ .

With several particles it becomes quite cumbersome to write out these hamiltonians explicitly. The lowest rank cases are

$$(4.26) \quad L_{\text{Ai}} = \partial^2 - x ; \quad \rho(t) = t^2 ,$$

$$(4.27) \quad L_{\text{Be}} = \partial^2 - x^{-2} ; \quad \rho(t) = t^2 - t - 1 .$$

Note that the first hamiltonian,  $h_1 = \text{tr}(\hat{Q})$ , is already non-linear. In the rank-two Bessel case with one particle, for instance,  $h_1(\lambda, \gamma) = \lambda^{-1}\rho(\lambda\gamma)$ . This gives the equations of motion

$$(4.28) \quad \dot{\lambda} = 8\gamma\lambda - 2$$

$$(4.29) \quad \dot{\gamma} = -4\gamma^2 - \lambda^{-2} .$$

These equations are solved by applying the Bessel involution and changing  $t$  to  $-t$ , i.e. by setting

$$(4.30) \quad \hat{\lambda} = c_1 ; \quad \hat{\gamma} = c_2 + t .$$



After some calculation,

$$(4.31) \quad \lambda(t) = 4c_1 t^2 + (8c_1 c_2 - 2)t - \frac{1}{c_1} - 2c_2 + 4c_1 c_2^2.$$

With two particles and a second order Airy operator, the first hamiltonian is

$$(4.32) \quad \gamma_1^2 + \gamma_2^2 - \lambda_1 - \lambda_2 - \frac{2}{(-\lambda_1 + \lambda_2)^2}.$$

With two particles and a second order Bessel operator, the first hamiltonian is

$$(4.33) \quad -\frac{\lambda_1 + \lambda_2}{\lambda_1 \lambda_2} - \gamma_1 + \lambda_1 \gamma_1^2 - \gamma_2 + \lambda_2 \gamma_2^2 + \frac{2(\lambda_1 \gamma_1 - \lambda_2 \gamma_2)}{-\lambda_1 + \lambda_2}.$$

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